Factorisation for parton distributions and related quantities

Part 2

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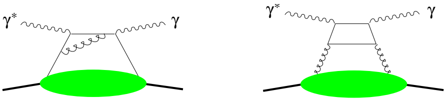
HELMHOLTZ

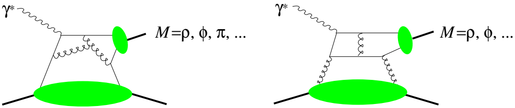
Key processes involving GPDs

- deeply virtual Compton scattering (DVCS)



also: $\gamma p \to \gamma^* p$ with $\gamma^* \to \ell^+ \ell^-$ (timelike CS)
$\gamma^* p \to \gamma^* p$ (double DVCS)

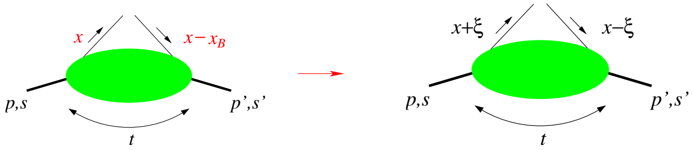
- meson production: large Q^2 or heavy quarks



involves also the distribution amplitude of the produced meson

- same factorisation concept, but for amplitude of exclusive processes rather than for (semi)inclusive cross section

GPDs: definition and properties



$$F^q = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-} \big\langle p', s' \big|\, \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\gamma^+ q(\tfrac{1}{2}z) \,\big|\, p, s \big\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

- kinematic variables:

  x, ξ momentum fractions w.r.t. $P = \frac{1}{2}(p + p')$

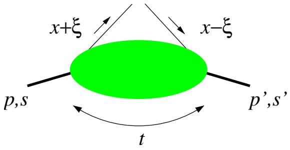
  $\xi = (p - p')^+/(p + p')^+$ plus-momentum transfer

  in DVCS: $\xi = x_B/(2 - x_B)$, x integrated over

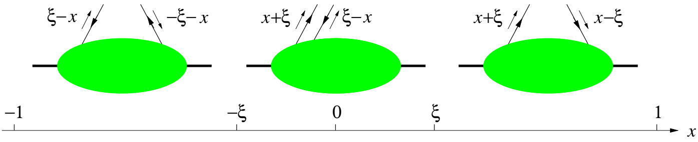
  t can trade for transverse momentum transfer $\boldsymbol{\Delta} = \boldsymbol{p}' - \boldsymbol{p}$

  $$t = -\frac{4\xi^2 m^2 + \boldsymbol{\Delta}^2}{1 - \xi^2}$$

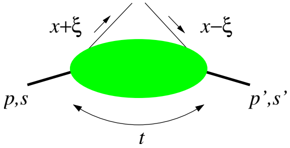
GPDs: definition and properties



- nonzero for $-1 \leq x \leq 1$
- $|x| > \xi$ similar to parton densities
  correlation $\psi^*_{x-\xi}\,\psi_{x+\xi}$ instead of probability $|\psi_x|^2$
  $|x| < \xi$ coherent emission of $q\bar{q}$ pair



GPDs: definition and properties



$$F^q = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-} \big\langle p', s' \big|\, \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\gamma^+ q(\tfrac{1}{2}z) \,\big|\, p, s \big\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

$$= H^q\, \bar{u}(p', s')\gamma^+ u(p, s) + E^q\, \bar{u}(p', s')\, \frac{i}{2m_p} \sigma^{+\alpha}(p'-p)_\alpha\, u(p, s)$$

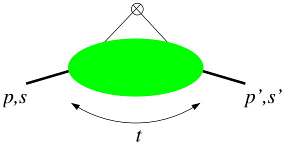
- proton spin structure:

  $H^q \leftrightarrow s = s'$ for $p = p'$ recover usual densities:

  $$H^q(x, \xi = 0, t = 0) = \begin{cases} q(x) & x > 0 \\ -\bar{q}(-x) & x < 0 \end{cases}$$

  $E^q \leftrightarrow s \neq s'$ decouples for $p = p'$
- similar definitions for polarised quarks $\tilde{H}^q, \tilde{E}^q$ and for gluons

GPDs: definition and properties



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$$= H^q\; \bar{u}(p', s')\gamma^+ u(p, s) + E^q\; \bar{u}(p', s')\, \frac{i}{2m_p} \sigma^{+\alpha}(p' - p)_\alpha\, u(p, s)$$

- Mellin moments: $\int dx\, x^n \;\to$ local operator $\to$ form factors
- can be calculated in lattice QCD
- $\int dx \;\to$ vector current $\bar{q}(0)\gamma^+ q(0)$

  $\sum_q e_q \int dx\, H^q(x, \xi, t) = F_1(t)$ Dirac f.f.

  $\sum_q e_q \int dx\, E^q(x, \xi, t) = F_2(t)$ Pauli f.f.
- $\int dx\, x \;\to$ energy-momentum tensor ⇝ lectures by Cédric Lorcé

Localising partons: impact parameter

- states with definite light-cone momentum p^+
  and transverse position (impact parameter):

$$|p^+, \boldsymbol{b}\rangle = \frac{1}{(2\pi)^2} \int d^2\boldsymbol{p}\, e^{-i\boldsymbol{b}\boldsymbol{p}}\, |p^+, \boldsymbol{p}\rangle$$

  formal: eigenstates of 2 dim. position operator

- can exactly localise proton in 2 dimensions
  no limitation by Compton wavelength

- and stay in frame where proton moves fast
  $\rightsquigarrow$ parton interpretation

- different from localisation in 3 spatial dimensions
  well-known for form factors; also for GPDs

Belitsky, Ji, Yuan '03; Brodsky et al. '06

Impact parameter GPDs

for simplicity take $\xi = 0$ and equal proton spins

- operator $\mathcal{O} = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-}\; \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z,\, \tfrac{1}{2}z]\, \gamma^+ q(\tfrac{1}{2}z)\big|_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$

$$
\begin{aligned}
&\langle p^+, -\boldsymbol{b}', s|\mathcal{O}|p^+, -\boldsymbol{b}, s\rangle \\
&\quad = (2\pi)^{-4} \int d^2\boldsymbol{p}' d^2\boldsymbol{p}\; e^{-i\boldsymbol{b}'\boldsymbol{p}' + i\boldsymbol{b}\boldsymbol{p}}\, \langle p^+, \boldsymbol{p}', s|\mathcal{O}|p^+, \boldsymbol{p}, s\rangle \\
&\quad = (2\pi)^{-4} \int d^2\boldsymbol{p}' d^2\boldsymbol{p}\; e^{-i\boldsymbol{b}'\boldsymbol{p}' + i\boldsymbol{b}\boldsymbol{p}}\, H_q(x, \xi = 0, t = -\boldsymbol{\Delta}^2) \\
&\quad = (2\pi)^{-4} \int d^2\boldsymbol{P}\; e^{-i(\boldsymbol{b}'-\boldsymbol{b})\boldsymbol{P}} \int d^2\boldsymbol{\Delta}\; e^{-i(\boldsymbol{b}'+\boldsymbol{b})\boldsymbol{\Delta}/2}\, H_q(x, \xi = 0, t = -\boldsymbol{\Delta}^2)
\end{aligned}
$$

with $P = \frac{1}{2}(p' + p)$ and $\Delta = p' - p$

$$
= \delta^{(2)}(\boldsymbol{b}' - \boldsymbol{b})\, (2\pi)^{-2} \int d^2\boldsymbol{\Delta}\; e^{-i\boldsymbol{b}\boldsymbol{\Delta}}\, H_q(x, \xi = 0, t = -\boldsymbol{\Delta}^2)
$$

Impact parameter GPDs

for simplicity take $\xi = 0$ and equal proton spins

- $q(x, b^2) = (2\pi)^{-2} \int d^2\boldsymbol{\Delta}\, e^{-i\boldsymbol{b}\boldsymbol{\Delta}}\, H^q(x, \xi = 0, t = -\boldsymbol{\Delta}^2)$
  gives distribution of quarks with
  - longitudinal momentum fraction x
  - transverse distance b from proton centre
- average impact parameter

$$\langle b^2 \rangle_x = \frac{\int d^2b\; b^2\, q(x, b^2)}{\int d^2b\; q(x, b^2)} = 4\, \frac{\partial}{\partial t} \log H(x, \xi = 0, t) \Big|_{t=0}$$

- integrated over x $\rightsquigarrow$ form factor

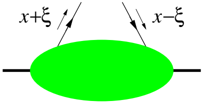
$$\langle b^2 \rangle = \frac{\int dx \int d^2b\; b^2\, q(x, b^2)}{\int dx \int d^2b\; q(x, b^2)} = 4\, \frac{\partial}{\partial t} \log F_1(t) \Big|_{t=0}$$

Evolution

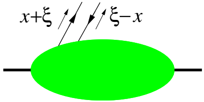
- for non-singlet combinations (e.g. $q - \bar{q}$ or $u - d$)

$$\mu^2 \frac{d}{d\mu^2} H^{\rm NS}(x,\xi,t) = \int dx' \, V^{\rm NS}(x,x',\xi) \, H^{\rm NS}(x',\xi,t)$$

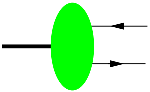
- for singlet $\sum_q (q + \bar{q})$: matrix equation for mixing with gluon GPD
- same evolution for E (independent of proton spin)



generalisation of DGLAP
evolution to $\xi \neq 0$
recover usual DGLAP for $\xi = 0$



ERBL evolution as for
meson distribution amplitudes (DAs)



- unifying treatment: evolution eqn. for operator $\mathcal{O}$, then take different matrix elements (PDFs, GPDs, DAs)

Evolution

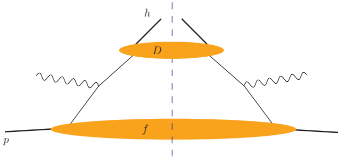
- for non-singlet combinations (e.g. $q - \bar{q}$ or $u - d$)

$$\mu^2 \frac{d}{d\mu^2} H^{\mathrm{NS}}(x, \xi, t) = \int dx' \, V^{\mathrm{NS}}(x, x', \xi) \, H^{\mathrm{NS}}(x', \xi, t)$$

- for singlet $\sum_q (q + \bar{q})$: matrix equation for mixing with gluon GPD
- same evolution for E (independent of proton spin)
- evolution local in t (take $-t \ll \mu^2$ to be safe)
  Fourier trf $\rightsquigarrow$ evolution local in b (take $1/\mu \ll b$ to be safe)
- for $\xi = 0$: $q(x, b^2)$ fulfils usual DGLAP evolution equation

$$\mu^2 \frac{d}{d\mu^2} q_{\mathrm{NS}}(x, b^2) = \int_x^1 \frac{dz}{z} \, P_{\mathrm{NS}}\Big(\frac{x}{z}\Big) q_{\mathrm{NS}}(z, b^2)$$

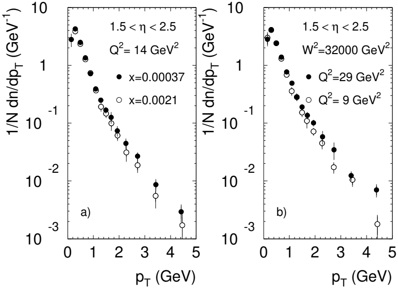
Measured transverse momentum



- consider
  - Drell-Yan with measured small q_T of γ^*
  - SIDIS with measured small q_T of hadron
  - $e^+e^- \to h_1 h_2 + X$ with h_1, h_2 approx. opposite momenta and small relative q_T
- $k_T \sim m$ from collinear graphs matters in final state
  - can still neglect parton k_T in hard scattering
  - but do not $\int d^2\boldsymbol{k}$ in parton densities and fragm. fcts.

    $\rightsquigarrow$ k_T dependent/unintegrated PDFs
    also called TMDs (transverse-momentum distributions)
- theoretical framework: TMD factorisation
  - also called k_T factorisation
    different from (but related to) k_T factorisation at small x

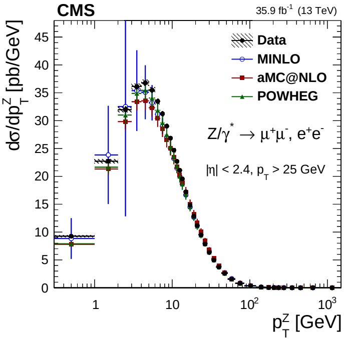
Measured transverse momentum



$$e + p \to e + h^{\pm} + X$$

$h^{\pm}$ = charged hadron

H1, hep-ex/9610006



$$p + p \to \ell^{+}\ell^{-} + X$$

CMS, arXiv:1909.04133

k_T dependent parton densities

k_T integrated:

$$f_1(x) = \int \frac{dz^-}{4\pi}\, e^{iz^- p^+ x} \left\langle p,s \middle| \bar{q}(0)\gamma^+ W(0,\infty) W(\infty,z^-)\, q(z^-) \middle| p,s \right\rangle \Big|_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

k_T dependent:

$$\int \frac{dz^-}{4\pi} \frac{d^2\boldsymbol{z}}{(2\pi)^2}\, e^{iz^- p^+ x}\, e^{-i\boldsymbol{k}\boldsymbol{z}} \left\langle p,s \middle| \bar{q}(0)\gamma^+ W(0,\infty) W(\infty,z^-,\boldsymbol{z})\, q(z^-,\boldsymbol{z}) \middle| p,s \right\rangle \Big|_{z^+=0}$$

- fields at different transv. positions
- notation: write k_T in text, omit $_T$ in boldface vector $\boldsymbol{k}$ etc.
- Fourier transform $\int d^2\boldsymbol{k}\; e^{i\boldsymbol{k}\boldsymbol{b}}$ of k_T dependent distribution gives

$$\int \frac{dz^-}{4\pi}\, e^{iz^- p^+ x} \left\langle p,s \middle| \bar{q}(0)\gamma^+ W(0,\infty) W(\infty,z^-,\boldsymbol{b})\, q(z^-,\boldsymbol{b}) \middle| p,s \right\rangle \Big|_{z^+=0}$$

note: $\boldsymbol{b}$ here not the same as in GPDs (will later call $\boldsymbol{z}$)

k_T dependent parton densities

k_T integrated:

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$$= f_1(x, \boldsymbol{k}^2) - \frac{\epsilon^{ij} \boldsymbol{k}^i \boldsymbol{s}^j}{m}\, f_{1T}^{\perp}(x, \boldsymbol{k}^2)$$

$$\epsilon^{12} = -\epsilon^{21} = 1 \qquad \epsilon^{11} = \epsilon^{22} = 0$$

- correlations between spins and transv. momentum
  e.g. Sivers function $f_{1T}^{\perp}$

A zoo of distributions

- collinear twist 2 densities:

  f_1 unpol. quark in unpol. proton
  g_1 correlate s_L of quark with S_L of proton
  h_1 correlate s_T of quark with S_T of proton

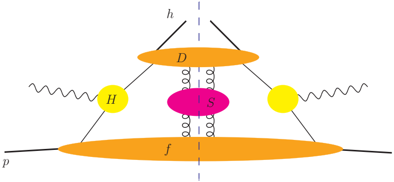
- k_T dependent twist 2 densities:

  f_1, g_1, h_1 as above
  $f_{1T}^{\perp}$ correlate k_T of quark with S_T of proton (Sivers)
  $h_1^{\perp}$ correlate k_T and s_T of quark (Boer-Mulders)
  $g_{1T}, h_{1T}^{\perp}, h_{1L}^{\perp}$ three more densities

- analogous for fragmentation functions:
  - $f_1 \leftrightarrow D_1$ unpolarized
  - $h_1^{\perp} \leftrightarrow H_1^{\perp}$ Collins fragm. fct.

TMD factorisation (SIDIS as example)

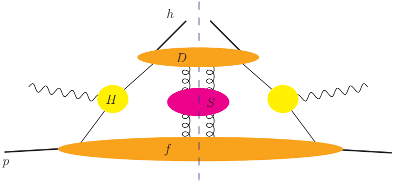
- take Q large and q_T small ($\sim m$ for power counting purposes)



- transverse-momentum dep't distribution and fragmentation fcts.
- only virtual corrections to hard subgraph
  no radiation of high-p_T partons allowed

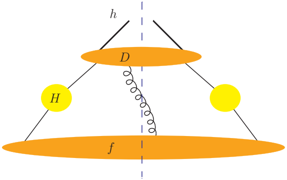
TMD factorisation (SIDIS as example)

- take Q large and q_T small ($\sim m$ for power counting purposes)



- soft gluon exchange does not cancel in sum over hadronic final state at leading-power accuracy gives soft factor in factorisation formula

$S =$ universal non-perturbative fct
$\rightarrow 1$ when integrate over k_T
cancellation between real



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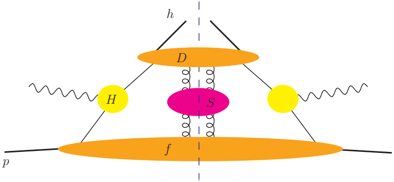

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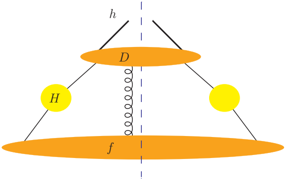
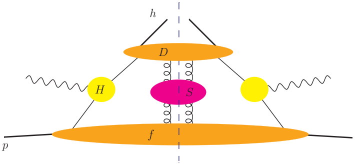
$\rightarrow 1$ when integrate over k_T
cancellation between real
and virtual graphs



SIDIS at low q_T

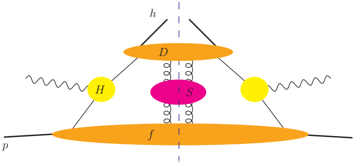


- factorisation formula

$$\frac{d\sigma_{\gamma^* p}}{dz\, d\boldsymbol{q}^2} = (\text{kin. fact.}) \times |H(\mu)|^2 \int d^2\boldsymbol{p}\; d^2\boldsymbol{k}\; d^2\boldsymbol{l}\; \delta^{(2)}\big(\boldsymbol{p} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{q}\big) \times \sum_{i=q,\bar{q}} e_i^2\, f_i(x, \boldsymbol{p}, \mu)\, D_i(z, \boldsymbol{k}, \mu)\, S(\boldsymbol{l}, \mu)$$

- various azimuthal and spin asymmetries
- no $\int d^2\boldsymbol{k}$ in parton densities $\rightsquigarrow$ no DGLAP type evolution !
  but μ dependence from renormalising UV divergences in virtual corrections

SIDIS at low q_T



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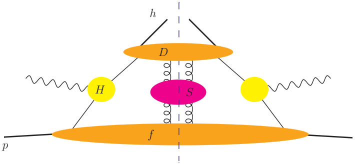
- simplifies if Fourier transform

$f(\boldsymbol{p}) \to f(\boldsymbol{b}),\ D(\boldsymbol{k}) \to D(\boldsymbol{b}),\ S(\boldsymbol{l}) \to S(\boldsymbol{b})$:

$$\frac{d\sigma_{\gamma^* p}}{dz\, d\boldsymbol{q}^2} = (\text{kin.fact.}) \times |H(\mu)|^2 \int d^2\boldsymbol{b}\; e^{-i\boldsymbol{b}\boldsymbol{q}} \sum_{i=q,\bar{q}} e_i^2\, f_i(x, \boldsymbol{b}, \mu)\, D_i(z, \boldsymbol{b}, \mu)\, S(\boldsymbol{b}, \mu)$$

SIDIS at low q_T



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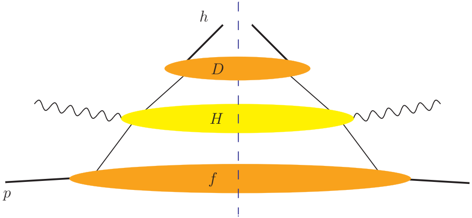
- simplifies if Fourier transform

  $f(\boldsymbol{p}) \to f(\boldsymbol{b})$, $D(\boldsymbol{k}) \to D(\boldsymbol{b})$, $S(\boldsymbol{l}) \to S(\boldsymbol{b})$:

$$\frac{d\sigma_{\gamma^* p}}{dz\, d\boldsymbol{q}^2} = (\text{kin.fact.}) \times |H(\mu)|^2 \int d^2\boldsymbol{b}\; e^{-i\boldsymbol{b}\boldsymbol{q}} \sum_{i=q,\bar{q}} e_i^2\, f_i(x,\boldsymbol{b},\mu)\, D_i(z,\boldsymbol{b},\mu)$$

- redefine f and D to each absorb factor $\sqrt{S}$
- evolution in a "rapidity parameter" $\rightsquigarrow$ Collins-Soper equation
  not discussed here for time reasons

Compare with collinear factorisation for large q_T



$$\frac{d\sigma_{\gamma^* p}}{dz\, d\boldsymbol{q}^2} = (\text{kin. fact.}) \times \int_x^1 \frac{d\hat{x}}{\hat{x}} \int_z^1 \frac{d\hat{z}}{\hat{z}}\, \delta\Big(\frac{\boldsymbol{q}^2}{Q^2} - \frac{(1-\hat{x})(1-\hat{z})}{\hat{x}\hat{z}} \Big) \times \sum_{i,j=q,\bar{q},g} f_i\Big(\frac{x}{\hat{x}}, \mu^2\Big) D_j\Big(\frac{z}{\hat{z}}, \mu^2\Big)\, C_{ij}\Big(\hat{x}, \hat{z}, \ln\frac{\mu^2}{Q^2}\Big)$$

- C_{ij} start at $\mathcal{O}(\alpha_s)$, must emit partons recoiling against $\boldsymbol{q}$
- convolution in momentum fractions

TMDs at large k_T

- for $k_T \gg m$ calculate k_T dependent densities from collinear ones:

$$f_1^i(x, \boldsymbol{k}^2; \zeta, \mu) = \frac{1}{\boldsymbol{k}^2} \sum_j \int_x^1 \frac{dx'}{x'} K^{ij}\Big(\frac{x}{x'}, \ln\frac{\boldsymbol{k}^2}{\zeta}\Big) f_1^j(x'; \mu)$$

K closely related with DGLAP splitting functions P

TMDs at large k_T

- for $k_T \gg m$ calculate k_T dependent densities from collinear ones:

$$f_1^i(x, \boldsymbol{b}^2; \zeta, \mu) = f_1^i(x; \mu) + \sum_j \int_x^1 \frac{dx'}{x'}\, \widetilde{K}^{ij}\Big(\frac{x}{x'}, \ln\frac{\mu^2}{\zeta}, \ln(\mu^2 \boldsymbol{b}^2)\Big)\, f_1^j(x'; \mu)$$

$\widetilde{K}$ closely related with DGLAP splitting functions P

- at operator level: expand $\bar{q}(0)\ldots q(z^-, \boldsymbol{b})$ around $\boldsymbol{b} = \boldsymbol{0}$
  a special case of the operator product expansion technique

Comparison between high-q_T and low-q_T descriptions

- collinear fact. requires $q_T \gg m$
  TMD fact. requires $q_T \ll Q$
  $\rightsquigarrow$ in region $m \ll q_T \ll Q$ both approaches are valid
- compare $q_T \gg m$ limit of TMD fact. result
  with $q_T \ll Q$ limit of coll. fact. result
  $\rightsquigarrow$ full agreement for unpol. cross section

  Collins, Soper, Sterman '85; Bacchetta et al. '08
- detailed comparison also for various spin asymmetries
  e.g. Sivers asy. in SIDIS or Drell-Yan at low q_T (Sivers fct.) and high q_T (Qiu-Sterman fct.)

  Ji, Qiu, Vogelsang, Yuan '06; Koike, Vogelsang, Yuan '07

Transverse momentum vs. position

- variables related by 2d Fourier transforms, e.g.
  - quark fields $\tilde{q}(\boldsymbol{k}, z^-) = \int d^2\boldsymbol{z}\, e^{i\boldsymbol{z}\boldsymbol{k}} q(\boldsymbol{z}, z^-)$
  - proton states $|p^+, \boldsymbol{b}\rangle = \int d^2\boldsymbol{p}\, e^{-i\boldsymbol{b}\boldsymbol{p}} |p^+, \boldsymbol{p}\rangle$
- in bilinear operators

$$\bar{\tilde{q}}(\boldsymbol{k})\, \tilde{q}(\boldsymbol{l}) = \int d^2\boldsymbol{y}\, d^2\boldsymbol{z}\, e^{-i(\boldsymbol{y}\boldsymbol{k} - \boldsymbol{z}\boldsymbol{l})}\, \bar{q}(\boldsymbol{y})\, q(\boldsymbol{z})$$

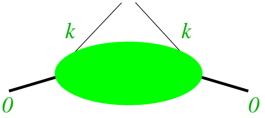
$$\boldsymbol{y}\boldsymbol{k} - \boldsymbol{z}\boldsymbol{l} = \tfrac{1}{2}(\boldsymbol{y} + \boldsymbol{z})(\boldsymbol{k} - \boldsymbol{l}) + \tfrac{1}{2}(\boldsymbol{y} - \boldsymbol{z})(\boldsymbol{k} + \boldsymbol{l})$$

'average' transv. momentum $\leftrightarrow$ position difference
transv. momentum transfer $\leftrightarrow$ 'average' position

- 'average' transv. mom. and position not Fourier conjugate

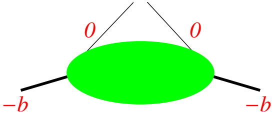
Mind the difference

TMDs



$\int d^2 \boldsymbol{z}\, e^{-i \boldsymbol{z} \boldsymbol{k}}\, \langle \mathbf{0} | \bar{q}(-\tfrac{1}{2}\boldsymbol{z}) \ldots q(\tfrac{1}{2}\boldsymbol{z}) | \mathbf{0} \rangle$

impact parameter distributions



$\int d^2 \boldsymbol{\Delta}\, e^{-i \boldsymbol{b} \boldsymbol{\Delta}}\, \langle -\tfrac{1}{2}\boldsymbol{\Delta} | \bar{q}(\mathbf{0}) \ldots q(\mathbf{0}) | \tfrac{1}{2}\boldsymbol{\Delta} \rangle$

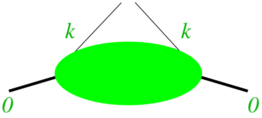
(longitudinal variables not shown for simplicity)

Fourier conjugates:

average transv. momentum $q(x, \boldsymbol{k})$	$\leftrightarrow$	difference of transv. positions Wilson lines, Sudakov resummation, ...
difference of transv. momenta $H(x, \boldsymbol{\Delta})_{\xi=0}$	$\leftrightarrow$	average transv. position $q(x, \boldsymbol{b})$

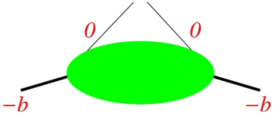
Mind the difference

TMDs



$$\int d^2 \boldsymbol{z}\, e^{-i \boldsymbol{z} \boldsymbol{k}} \, \langle \mathbf{0} | \bar{q}(-\tfrac{1}{2}\boldsymbol{z}) \dots q(\tfrac{1}{2}\boldsymbol{z}) | \mathbf{0} \rangle$$

impact parameter distributions



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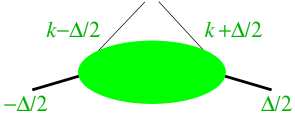
more general:

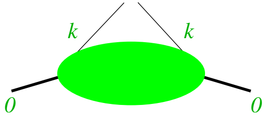
GTMDs



$$\int d^2 \boldsymbol{z}\, e^{-i \boldsymbol{z} \boldsymbol{k}} \, \langle -\tfrac{1}{2}\boldsymbol{\Delta} | \bar{q}(-\tfrac{1}{2}\boldsymbol{z}) \dots q(\tfrac{1}{2}\boldsymbol{z}) | \tfrac{1}{2}\boldsymbol{\Delta} \rangle$$

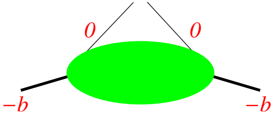
Mind the difference

TMDs



$\int d^2\boldsymbol{z}\, e^{-i\boldsymbol{z}\boldsymbol{k}}\, \langle \boldsymbol{0}|\bar{q}(-\tfrac{1}{2}\boldsymbol{z}) \ldots q(\tfrac{1}{2}\boldsymbol{z})|\boldsymbol{0}\rangle$

impact parameter distributions



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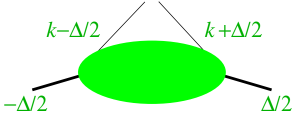
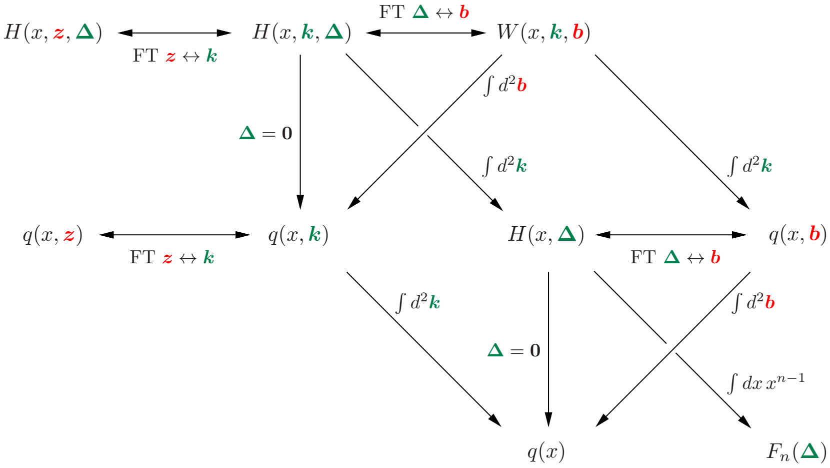


$\int d^2\boldsymbol{z}\, e^{-i\boldsymbol{z}\boldsymbol{k}}\, \langle -\tfrac{1}{2}\boldsymbol{\Delta}|\bar{q}(-\tfrac{1}{2}\boldsymbol{z}) \ldots q(\tfrac{1}{2}\boldsymbol{z})|\tfrac{1}{2}\boldsymbol{\Delta}\rangle$

Fourier transf. from $\boldsymbol{\Delta}$ to $\boldsymbol{b}$
$\rightsquigarrow$ Wigner functions

parton momentum and position
within limits of uncertainty rel'n

Relations



GPDs taken at zero skewness $\xi = 0$

$\int d^2\boldsymbol{k}$ needs UV regularisation

$\boldsymbol{k}$ dep't functions have ζ dep'ce

Integration over k_T

- naive:

$$\int d^2\boldsymbol{k}\, q(x,\boldsymbol{k}) = q(x)$$

cannot be true because $q(x,\boldsymbol{k}) \sim 1/\boldsymbol{k}^2$ at large $\boldsymbol{k}$

- correct:

$$\int_{\boldsymbol{k}^2<\mu^2} d^2\boldsymbol{k}\, q(x,\boldsymbol{k};\mu) = q(x;\mu) + \text{calculable terms of } \mathcal{O}(\alpha_s)$$

- Fourier trf. from $\boldsymbol{k}$ to $\boldsymbol{z}$:

instead of $\int_{\boldsymbol{k}^2<\mu^2} d^2\boldsymbol{k}$ take $\int d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\boldsymbol{z}}$ with $|\boldsymbol{z}| = 1/\mu$

oscillations suppress region $|\boldsymbol{k}| \gg 1/|\boldsymbol{z}|$

$$q(x,\boldsymbol{z};\mu) = q(x;\mu) + \text{calculable terms of } \mathcal{O}(\alpha_s)$$

see earlier slide

Summary of part 2

- GPDs
  - extend factorisation concept to exclusive processes
  - factorisation of amplitude instead of cross section
  - gives access to transverse spatial distribution of partons
- TMD factorisation for measured $q_T \ll$ hard scale
  - important differences with collinear factorisation, different evolution
  - valid for restricted class of processes
    for some cases smooth theoretical transition to high q_T regime
  - theoretically controlled access to transverse parton momentum
- Wigner functions: unifying framework for describing transverse momentum and position

The operator product expansion in a nutshell

- consider product of operators $\mathcal{O}_1(0)\,\mathcal{O}_2(z)$ for $z^\mu \to 0$
- operator product expansion:

$$\mathcal{O}_1(0)\,\mathcal{O}_2(z) \approx \sum_i C_i(z^2)\,\theta_i(0)$$

  - $C_i(z^2)$ = coefficient functions
    $\theta_i(z)$ = local operators allowed by quantum numbers
  - dimensional analysis:

$$C_i(z^2) \sim |z|^{\dim(\theta_i)-\dim(\mathcal{O}_1)-\dim(\mathcal{O}_2)} \qquad |z| = \sqrt{|z^2|}$$

    up to powers of $\log \mu|z|$ from renormalization
    $\rightsquigarrow$ operators θ_i with lowest mass dimension dominate for $|z| \to 0$
- provides a systematic way to include power corrections suppressed by $|z|^p$

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    $\rightsquigarrow$ operators θ_i with lowest mass dimension dominate for $|z| \to 0$
- provides a systematic way to include power corrections suppressed by $|z|^p$
- can generalise to operators $\mathcal{O}_1, \mathcal{O}_2$ with Lorentz indices
  local operators are e.g.

$\bar{q}\gamma^+ \big[\overleftrightarrow{D}^+\big]^n q$	twist 2
$\bar{q}\gamma^+ \big[\overleftrightarrow{D}^+\big]^{n_1} F^{+i} \big[\overleftrightarrow{D}^+\big]^{n_2} q$	twist 3

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    up to powers of $\log \mu|z|$ from renormalization
    $\rightsquigarrow$ operators θ_i with lowest mass dimension dominate for $|z| \to 0$

- provides a systematic way to include power corrections suppressed by $|z|^p$
- can generalise to limit $z^2 \to 0$ of light-like distances
  $\rightsquigarrow$ expand on non-local operators $\theta_i(\hat{z})$ with $\hat{z}^2 = 0$

The operator product expansion in a nutshell

- examples for light-cone expansion

$$\mathcal{O}_1(0)\,\mathcal{O}_2(z) \approx \sum_i C_i(z^2)\,\theta_i(\hat{z}) \qquad\qquad \hat{z}^2 = 0$$

- TMD at small b
  - expand $[\bar{q}(0)\gamma^+ \ldots q(z)]_{z^+=0,\boldsymbol{z}=\boldsymbol{b}}$ for $z^2 = -\boldsymbol{b}^2 \to 0$
  - leading light-cone operator: $[\bar{q}(0)\gamma^+ \ldots q(\hat{z})]_{\hat{z}^+=0,\hat{\boldsymbol{z}}=\boldsymbol{0}}$
  - proton matrix element $\rightsquigarrow$ relate TMD at small b with PDF

The operator product expansion in a nutshell

- examples for light-cone expansion

$$\mathcal{O}_1(0)\,\mathcal{O}_2(z) \approx \sum\nolimits_i C_i(z^2)\,\theta_i(\hat{z}) \qquad\qquad \hat{z}^2 = 0$$

- TMD at small b
  - expand $[\bar{q}(0)\gamma^+ \ldots q(z)]_{z^+=0,\boldsymbol{z}=\boldsymbol{b}}$ for $z^2 = -\boldsymbol{b}^2 \to 0$
  - leading light-cone operator: $[\bar{q}(0)\gamma^+ \ldots q(\hat{z})]_{\hat{z}^+=0,\hat{\boldsymbol{z}}=\boldsymbol{0}}$
  - proton matrix element $\rightsquigarrow$ relate TMD at small b with PDF
- Compton tensor for DIS (see lect. 3 by K Golec-Biernat, p.3) or for DVCS
  - prod. of two electromagnetic currents $J^\mu(0)\,J^\nu(z)$ or $T\,J^\mu(0)\,J^\nu(z)$ dominated by $z^2 \to 0$ in Bjorken limit (must show)
  - leading light-cone operator: $[\bar{q}(0)\gamma^+ \ldots q(\hat{z})]_{\hat{z}^+=0,\hat{\boldsymbol{z}}=\boldsymbol{0}}$
  - proton matrix elements
    $\rightsquigarrow$ Compton amplitude in terms of PDFs or GPDs
  - $|z|^p$ corrections in operator product $\rightsquigarrow$ Q^{-p} corr.s in matrix element
  - allows e.g. treatment of target mass corrections $(m/Q)^p$
    O Nachtmann, 1973 for DIS; V Braun et al $\geq$ 2012 for DVCS