Factorisation for parton distributions and related quantities

Part 1

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EICPL lecture series on QCD for EIC
Online, 4 April 2022

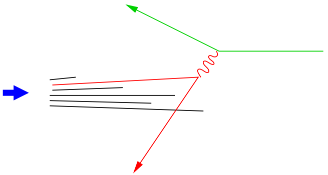
Plan of the lectures

- today
  - collinear factorisation
  - parton distributions and fragmentation functions
- next week
  - generalised parton distributions
  - transverse-momentum distributions
  - three-dimensional imaging of hadrons
- in general, focus on the theoretical ideas
  and show some simple calculations
  to present the associated phenomenology would require $\gg$ time

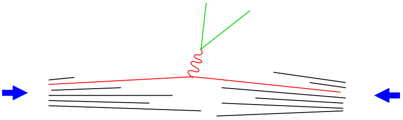
The parton model

description for deep inelastic scattering, Drell-Yan process, etc.

- fast-moving hadron
  $\approx$ set of free partons $(q, \bar{q}, g)$ with low transverse momenta
- physical cross section
  $=$ cross section for partonic process $(\gamma^* q \to q,\ q\bar{q} \to \gamma^*)$
  $\times$ parton densities



Deep inelastic scattering (DIS): $\ell p \to \ell X$



Drell-Yan: $pp \to \ell^+ \ell^- X$

Nobel prize 1990 for
Friedman, Kendall, Taylor

The parton model

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Factorisation

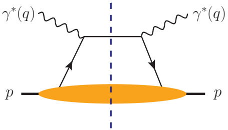
- implement and correct parton-model ideas in QCD
  - conditions and limitations of validity
    kinematics, processes, observables
  - corrections: partons interact
    α_s is small at large scales $\rightsquigarrow$ perturbation theory
  - definition of parton densities in QCD
    derive their general properties
    make contact with non-perturbative methods

Example: inclusive DIS (deep inelastic scattering)

- $\sigma_{\text{tot}}(\gamma^* p \to X)$

  $\xrightarrow{\text{opt. theorem}}$ $\operatorname{Im} \mathcal{A}(\gamma^* p \to \gamma^* p)$

  forward amplitude



- measure in $ep \to eX$
- Bjorken limit: $Q^2 = -q^2 \to \infty$ at fixed $x_B = Q^2/(2p \cdot q)$
- $\operatorname{Im} \mathcal{A}(\gamma^* p \to \gamma^* p) =$

  hard-scattering coefficient $\otimes$ parton distribution

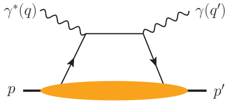
  - hard-scattering coefficient $\sim$ $\operatorname{Im} \mathcal{A}(\gamma^* q \to \gamma^* q)$ small print $\to$ later
  - parton densities (PDFs): process independent
    also appear in $pp \to \ell^+ \ell^- X$, $\gamma^* p \to \text{jet} + X$, ...

Example: DVCS (deeply virtual Compton scattering)

- exclusive cross section

  $\propto \left|\mathcal{A}(\gamma^* p \to \gamma p)\right|^2$

  square of amplitude



- measure in $ep \to ep\gamma$
- Bjorken limit: $Q^2 = -q^2 \to \infty$ at fixed x_B and $t = (p - p')^2$
- $\mathcal{A}(\gamma^* p \to \gamma p) =$

  hard-scattering coefficient $\otimes$ generalised parton distribution

  - GPD depends on momentum fractions x, ξ and on t
  - hard-scattering coefficient $\sim$ $\mathcal{A}(\gamma^* q \to \gamma q)$

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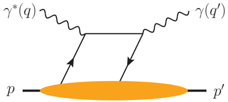

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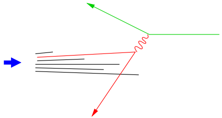
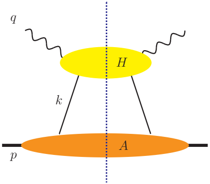
  hard-scattering coefficient $\otimes$ generalised parton distribution

  - GPD depends on momentum fractions x, ξ and on t
  - hard-scattering coefficient $\sim$ $\mathcal{A}(\gamma^* q \to \gamma q)$ or $\mathcal{A}(\gamma^* q\bar{q} \to \gamma)$
    both cases included in "$\otimes$" of factorisation formula

Factorisation: physics idea and technical implementation

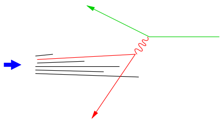


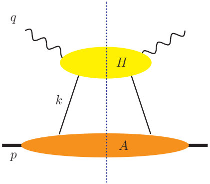


- idea: separation of physics at different scales
  - high scales: quark-gluon interactions
    $\rightsquigarrow$ compute in perturbation theory
  - low scale: proton $\rightarrow$ quarks, antiquarks, gluons
    $\rightsquigarrow$ parton densities

requires hard momentum scale in process
large photon virtuality $Q^2 = -q^2$ in DIS

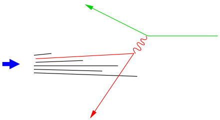
Factorisation: physics idea and technical implementation

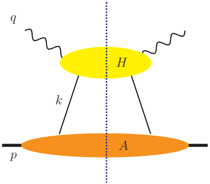




- implementation: separate process into
  - "hard" subgraph H with particles far off-shell
    compute in perturbation theory
  - "collinear" subgraph A with particles moving along proton
    turn into definition of parton density

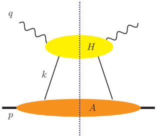
Factorisation: physics idea and technical implementation





- note difference with high-energy/small x factorisation
  - separate dynamics according to rapidity (not virtuality) of particles
  - overlap of the factorisation schemes if have strong ordering in rapidity and virtuality

Collinear expansion



- graph gives $\int d^4k\, H(k) A(k)$; simplify further
- light-cone coordinates: $v^{\pm} = \frac{1}{\sqrt{2}}(v^0 \pm v^3)$, $\boldsymbol{v} = (v^1, v^2)$

$$v^\mu w_\mu = v^+ w^- + v^- w^+ - \boldsymbol{v}\boldsymbol{w} \qquad v^2 = 2v^+ v^- - \boldsymbol{v}^2$$

$$d^4v = dv^+\, dv^-\, d^2\boldsymbol{v}$$

boost along z: $\quad v^+ \to \alpha v^+ \qquad v^- \to v^-/\alpha \qquad \boldsymbol{v} \to \boldsymbol{v}$

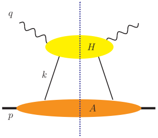
for fast right-moving particle $(p^3 \gg |\boldsymbol{p}|, m)$ have

$$p^0 = \sqrt{(p^3)^2 + \boldsymbol{p}^2 + m^2} \approx p^3 + \frac{\boldsymbol{p}^2 + m^2}{2p^3}$$

$$\Rightarrow \quad \sqrt{2}p^+ = p^0 + p^3 \approx 2p^3$$

$$\sqrt{2}p^- = p^0 - p^3 \approx \frac{\boldsymbol{p}^2 + m^2}{2p^3} \ll |\boldsymbol{p}|, m$$

Collinear expansion



- graph gives $\int d^4k\, H(k) A(k)$; simplify further
- in hard graph neglect small components of external lines
  $\rightsquigarrow$ Taylor expansion

$$H(k^+, k^-, k_T) = H(k^+, 0, 0) + \text{corrections}$$

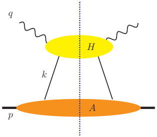
  $\rightsquigarrow$ loop integration greatly simplifies:

$$\int d^4k\ H(k)\, A(k) \approx \int dk^+\, H(k^+, 0, 0)\ \int dk^- d^2k_T\, A(k^+, k^-, k_T)$$

- in hard scattering treat incoming/outgoing partons as exactly collinear $(k_T = 0)$ and on-shell $(k^- = 0)$
- in collin. matrix element integrate over k_T and virtuality
  $\rightsquigarrow$ collinear (or k_T integrated) parton densities only depend on $k^+ = xp^+$

further subtleties related with spin of partons, not discussed here

Definition of parton distributions



- matrix elements of quark/gluon operators

$$f(x) = \int \frac{dz^-}{2\pi}\, e^{ixp^+z^-} \left\langle p \middle| \bar{q}(0)\, \tfrac{1}{2}\gamma^+ W[0,z]\, q(z) \middle| p \right\rangle \Big|_{z^+=0,\, z_T=0}$$

$q(z)$ = quark field operator: annihilates quark

$\bar{q}(0)$ = conjugate field operator: creates quark

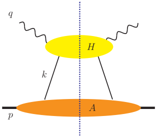
$\frac{1}{2}\gamma^+$ sums over quark spin

$\int \frac{dz^-}{2\pi}\, e^{ixp^+z^-}$ projects on quarks with $k^+ = xp^+$

$W[0,z]$ = Wilson line, makes product of fields gauge invariant
in light-cone gauge $A^+ = 0$ have $W[0,z] = 1$

- analogous definitions for polarised quarks, antiquarks, gluons
- analysis of factorisation used Feynman graphs
  but here provide non-perturbative definition

Definition of parton distributions



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$q(z) =$ quark field operator: annihilates quark and creates antiquark

$\bar{q}(0) =$ conjugate field operator: creates quark and annihilates antiquark

- matrix element generates both quark and antiquark distributions:

$$f(x) = \begin{cases} q(x) & \text{for } x > 0, \\ -\bar{q}(-x) & \text{for } x < 0 \end{cases}$$

minus sign from $dd^\dagger = -d^\dagger d$

Lowest order results for DIS and DVCS

- hard-scattering part of handbag graphs:
  take Bjorken limit and frame with $\boldsymbol{p} = \boldsymbol{q} = \boldsymbol{0}$

$$Q^2 = -q^2 = -2q^+q^- \,, \qquad m^2 = 2p^+p^-$$

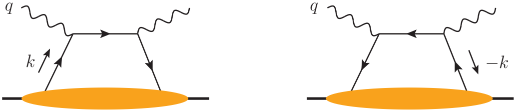
$$\frac{Q^2}{x_B} = 2pq = 2p^+q^- + 2p^-q^+ = -\frac{p^+}{q^+}\,Q^2 + \frac{q^+}{p^+}\,m^2 \approx -\frac{p^+}{q^+}\,Q^2$$

$$\Rightarrow \quad q^+ \approx -x_B\,p^+ \,, \qquad q^- \approx \frac{Q^2}{2x_B\,p^+}$$

Breit frame: $q^0 = 0 \qquad \Rightarrow \quad q^- = -q^+ = Q/\sqrt{2}$

$$\Rightarrow \quad p^+ \approx \frac{Q}{\sqrt{2}x_B} \,, \quad p^- \approx \frac{x_B m^2}{\sqrt{2}Q} \ll m$$

Lowest order results for DIS and DVCS



- hard-scattering part of handbag graphs:
  take Bjorken limit and frame with $\boldsymbol{p} = \boldsymbol{q} = \boldsymbol{0}$
  define $x = k^+/p^+ \approx k^3/p^3$

  propagator denominator (left graph):

$$
\begin{aligned}
(k+q)^2 + i\epsilon &= 2(k^- + q^-)(k^+ + q^+) - \boldsymbol{k}^2 + i\epsilon \\
&\approx 2q^-(k^+ + q^+) + i\epsilon \\
&\approx 2q^-(xp^+ - x_B p^+) + i\epsilon \\
&\approx \frac{Q^2}{x_B}(x - x_B) + i\epsilon
\end{aligned}
$$

Lowest order results for DIS and DVCS

- hard-scattering part of handbag graphs:
  take Bjorken limit and frame with $\boldsymbol{p} = \boldsymbol{q} = \boldsymbol{0}$

$$\frac{1}{x - x_B + i\varepsilon} + \{\text{crossed graph}\} = \mathrm{PV}\,\frac{1}{x - x_B} - i\pi\delta(x - x_B) + \{\text{crossed graph}\}$$

- for DIS:

$$\sigma_{\text{tot}} \propto \mathrm{Im}\,\mathcal{A}(\gamma_T^* p \to \gamma_T^* p) = \sum_q (e e_q)^2 \big[q(x_B) + \bar{q}(x_B) \big]$$

$$\mathcal{A}(\gamma_L^* p \to \gamma_L^* p) = 0$$

$$2x_B F_1 = F_2 = x_B \sum_q e_q^2 \big[q(x_B) + \bar{q}(x_B) \big]$$

Lowest order results for DIS and DVCS

- hard-scattering part of handbag graphs:
  take Bjorken limit and frame with $\boldsymbol{p} = \boldsymbol{q} = \boldsymbol{0}$

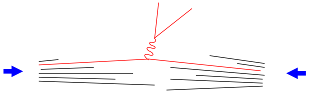
$$\frac{1}{x - x_B + i\varepsilon} + \{\text{crossed graph}\} = \mathrm{PV}\, \frac{1}{x - x_B} - i\pi\delta(x - x_B) + \{\text{crossed graph}\}$$

- for DVCS:

$$\mathcal{A}(\gamma_T^* p \to \gamma_T p) = \sum_q (e e_q)^2 \left[\mathrm{PV} \int dx\, \frac{\mathsf{GPD}(x, x_B, t)}{x_B - x} + i\pi\, \mathsf{GPD}(x_B, x_B, t) \right] + \{\text{crossed graph}\}$$

Factorisation for pp collisions

- example: Drell-Yan process $pp \to \gamma^* + X \to \mu^+\mu^- + X$ where $X =$ any number of hadrons
- one parton distribution for each proton $\times$ hard scattering $\rightsquigarrow$ deceptively simple physical picture



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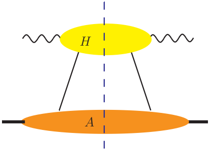
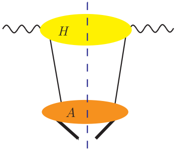
- "spectator" interactions produce additional particles
  which are also part of unobserved system X ("underlying event")
- need not calculate this thanks to unitarity
  as long as cross section/observable sufficiently inclusive
- but must calculate/model if want more detail on the final state
  $\rightsquigarrow$ factorisation does not work for all observables

Fragmentation

- cross DIS $eh \to e + X$ to $e^+e^- \to \bar{h} + X$
  i.e., $\gamma^* h \to X$ to $\gamma^* \to \bar{h} + X$





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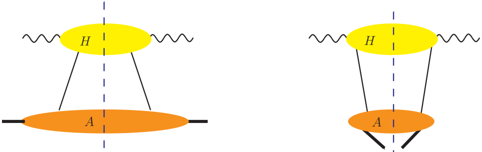
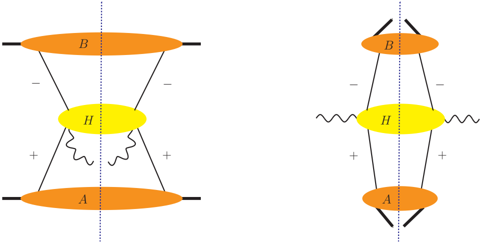


- or Drell-Yan $h_1 h_2 \to \gamma^* + X$ to $\gamma^* \to \bar{h}_1 \bar{h}_2 + X$



Fragmentation

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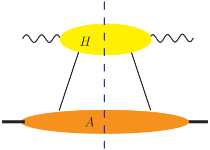
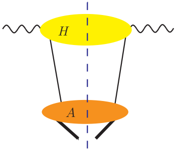
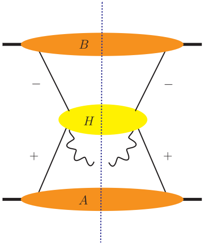
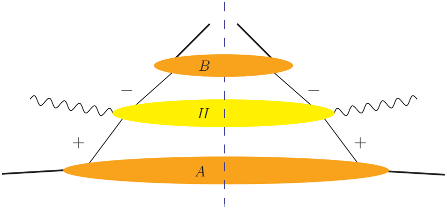




- or Drell-Yan $h_1 h_2 \to \gamma^* + X$ to SIDIS $\gamma^* h_1 \to \bar{h}_2 + X$





Fragmentation functions

- replace parton density $k^+ = xp^+$

$$f(x) = \int \frac{d\xi^-}{4\pi}\, e^{i\xi^- p^+ x} \langle h|\bar{q}(0)\gamma^+ W(0,\xi^-)q(\xi^-)|h\rangle$$

$$= \sum_X \int \frac{d\xi^-}{4\pi}\, e^{i\xi^- p^+ x}$$

$$\times \sum_X \big\langle h\big|(\bar{q}(0)\gamma^+)_\alpha W(0,\infty)\big|X\big\rangle\, \big\langle X\big|W(\infty,\xi^-)q_\alpha(\xi^-)\big|h\big\rangle$$

by fragmentation function $p^+ = zk^+$

$$D(z) = \frac{1}{2N_c z} \int \frac{d\xi^-}{4\pi}\, e^{i\xi^- p^+/z}$$

$$\times \sum_X \big\langle 0\big|W(\infty,\xi^-)q_\alpha(\xi^-)\big|\bar{h}X\big\rangle\, \big\langle \bar{h}X\big\rangle\big|(\bar{q}(0)\gamma^+)_\alpha W(0,\infty)\big|0\big\rangle$$

$N_c = 3$ number of colours

A closer look at one-loop corrections

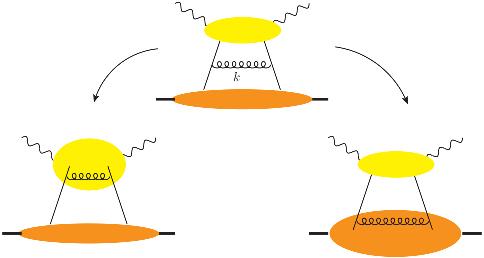
- example: DIS

- UV divergences removed by standard renormalisation
- soft divergences cancel in sum over graphs
- collinear div. do not cancel, have integrals

$$\int_0 \frac{dk_T^2}{k_T^2}$$

what went wrong?

- hard graph should not contain internal collinear lines
  collinear graph should not contain hard lines
- must not double count $\rightsquigarrow$ factorisation scale μ



- with cutoff: take $k_T > \mu$ take $k_T < \mu$
  $1/\mu \sim$ transverse resolution

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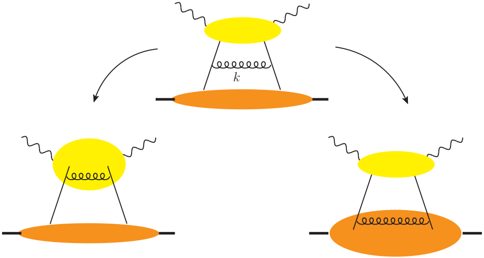
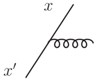


- with cutoff: take $k_T > \mu$ — take $k_T < \mu$
  $1/\mu \sim$ transverse resolution
- in dim. reg.:
  subtract collinear divergence — subtract ultraviolet div.

The evolution equations

- DGLAP equations

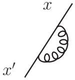
$$\frac{d}{d\log\mu^2} f(x,\mu) = \int_x^1 \frac{dx'}{x'} P\Big(\frac{x}{x'}\Big) f(x',\mu) \;=\; \big(P \otimes f(\mu)\big)(x)$$



- $P =$ splitting functions
  - have perturbative expansion

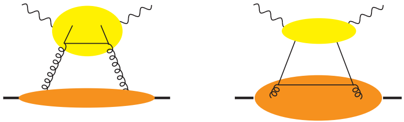
$$P(x) = \alpha_s(\mu)\, P^{(0)}(x) + \alpha_s^2(\mu)\, P^{(1)}(x) + \alpha_s^3(\mu)\, P^{(2)}(x) \ldots$$

    fully known to 3 loops, partially to 4 loops
    Moch et al. 2004, 2017



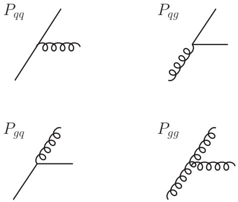
  - contains terms $\propto \delta(1-x)$ from virtual corrections

- quark and gluon densities mix under evolution:



- matrix evolution equation

$$\frac{d}{d\log\mu^2} f_i(x,\mu) = \sum_{j=q,\bar{q},g} \big(P_{ij} \otimes f_j(\mu)\big)(x) \qquad (i,j=q,\bar{q},g)$$



- parton content of proton depends on resolution scale μ

Mellin moments

- Mellin moments: $M(j) = \int_0^1 dx\, x^{j-1}\, f(x)$
- anomalous dimensions: $\gamma(j) = \int_0^1 dx\, x^{j-1}\, P(x)$

$$\frac{d f(x)}{d \log \mu^2} = (P \otimes f)(x) \quad \Rightarrow \quad \frac{dM(j)}{d \log \mu^2} = \gamma(j)\, M(j)$$

Mellin moments

- Mellin moments: $M(j) = \int_0^1 dx\, x^{j-1}\, f(x)$
- matrix element definition

$$f(x) = \int \frac{dz^-}{2\pi}\, e^{ixp^+z^-}\, \langle p \,|\, \bar{q}(-\tfrac{1}{2}z)\, \tfrac{1}{2}\gamma^+ W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\, q(\tfrac{1}{2}z) \,|\, p \rangle \Big|_{z^+=0,\, z_T=0}$$

operator identity (simple to show in gauge $A^+ = 0$, where $W[...] = 1$)

$$\int dx\, x^{n-1} \int dz^-\; e^{ixp^+z^-}\; \bar{q}(-\tfrac{1}{2}z)\gamma^+ W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\, q(\tfrac{1}{2}z)\Big|_{z^+=0,\, z_T=0}$$
$$\propto\; \bar{q}(z)\gamma^+ \big[\overleftrightarrow{D}^+(z)\big]^{n-1}\, q(z)\,\Big|_{z=0}$$

with $\overleftrightarrow{D} = \frac{1}{2}(\overrightarrow{D} - \overleftarrow{D})\,, \qquad \overrightarrow{D}(z) = \partial_z + igA(z)\,, \qquad \overleftarrow{D}(z) = \overleftarrow{\partial}_z - igA(z)$

$\Rightarrow \quad \int_{-1}^{1} dx\, x^{n-1}\, f(x) =$ matrix element of a local operator

$$= \begin{cases} \int_0^1 dx\, x^{n-1}\, [q(x) - \bar{q}(x)] & \text{for odd } n \\ \int_0^1 dx\, x^{n-1}\, [q(x) + \bar{q}(x)] & \text{for even } n \end{cases}$$

Factorisation formulae

- example: SIDIS $e + p \to h + X$

$$\sigma(e + p \to h + X) = \sum_{i,j=q,\bar{q},g} \int dx\, dz\; f_i(x, \mu_F)\, D_j(z, \mu_F) \times \hat{\sigma}_{ij}\big(x, z, \alpha_s(\mu_R), \mu_R, \mu_F, Q^2\big) + \mathcal{O}\left(\frac{\Lambda^2}{Q^4}\right)$$

  - $\hat{\sigma}_{ij}$ = cross section for hard scattering $\gamma^* + i \to j + X$
    Q^2 provides hard scale
  - μ_R = renormalisation scale, μ_F = factorisation scale
    may take different or equal
  - μ_F dependence in $\hat{\sigma}$, f, D cancels up to higher orders in α_s
    similar discussion as for μ_R dependence
  - accuracy: α_s expansion and power corrections $\mathcal{O}(\Lambda^2/Q^2)$
- can make σ and $\hat{\sigma}$ differential, e.g. in x_B, p_T of h, ...

Summary of part 1

Factorisation

- implements ideas of parton model in QCD
  - perturbative corrections (NLO, NNLO, N^3LO, ...)
  - field theoretical def. of parton densities
    $\rightsquigarrow$ bridge to non-perturbative QCD
- valid for sufficiently inclusive observables
  and up to power corrections in Λ/Q or $(\Lambda/Q)^2$
  which are most often not calculable
- must in a consistent way
  - remove collinear kinematic region in hard scattering
  - remove hard kinematic region in parton densities
    $\leftrightarrow$ UV renormalisation

  this introduces factorisation scale μ_F
  - separates "collinear" from "hard", "object" from "probe"